وزارة التعليم العالي والبحث العلمي
المعهد التقني/ السماوة
قسم المساحة

اسم المادة : Mathematics and Spherical Triangles

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- Determinant
- Matrix inverse

Determinant

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just basic arithmetic. Here is how:

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc)

Example: **Find the determinant of the following matrix.**

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$|B| = 4\times8 - 6\times3$$

$$= 32 - 18$$

$$= 14$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\left[a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \right] - \left[b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \right] + \left[c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right]$$

To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not in a**'s row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The determinant is:

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

"The determinant of A equals ... etc"

Example: Find the determinant of the 3×3 matrix.

$$|A| = \begin{vmatrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 6 & -2 \\ 2 & -3 \end{vmatrix} - (2) \cdot \begin{vmatrix} 5 & -2 \\ 5 & -3 \end{vmatrix} + (-4) \cdot \begin{vmatrix} 5 & 6 \\ 5 & 2 \end{vmatrix}$$

$$= 6\left[-18 - (-4)\right] - 2\left[-15 - (-10)\right] - 4\left[10 - 30\right]$$

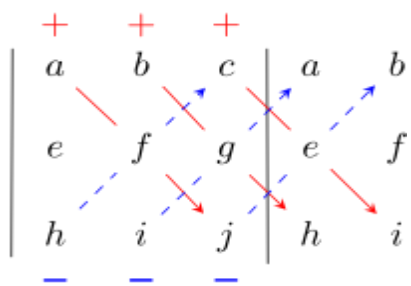
$$= 6(-14) - 2(-5) - 4(-20)$$

$$= -84 + 10 + 80$$

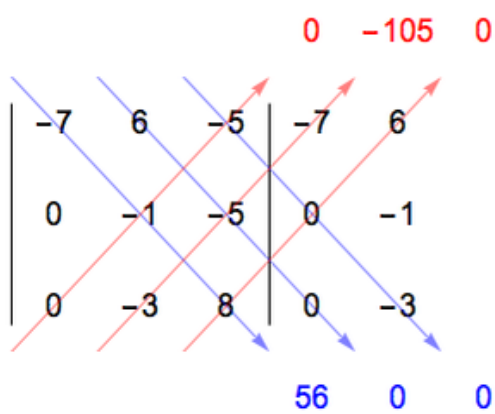
$$= 6$$

3×3 Determinants Using Diagonals

The rule of Diagonals for the 3 × 3 matrix determinant: the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration:



Example 1: Find the determinant of the matrix.



$$\therefore \begin{vmatrix} -7 & 6 & -5 \\ 0 & -1 & -5 \\ 0 & -3 & 8 \end{vmatrix} = 56 + 0 + 0 - (0 + -105 + 0) = 161$$

Matrix inverse

If a 2×2 matrix A is invertible and is multiplied by its inverse (denoted by the symbol $\mathbf{A}^{-1}$), the resulting product is the Identity matrix which is denoted by I.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its inverse is calculated using the formula

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1: Find the inverse of the 2×2 matrix below, if it exists.

$$A = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}$$

The formula requires us to find the determinant of the given matrix, the formula below how to solve for the determinant of a 2×2 matrix.

$$\text{Let matrix } A \text{ be } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } \det A = ad - bc$$

Always subtract!

So then, the determinant of matrix A is

$$\det A = \det\begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix} = (5)(-3) - (2)(-7)$$

$$= (-15) - (-14)$$

$$= -15 + 14$$

$$\det A = -1$$

So then, the determinant of matrix A is

- Here goes again the formula to find the inverse of a 2×2 matrix.

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Now, let's find the inverse of matrix A.

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}$$

$$= -1\begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) & (-1)(-2) \\ (-1)(7) & (-1)(5) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix}$$

Example 2: Find the inverse of the 2×2 matrix below, if it exists.

$$B = \begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$$

Solution :

First, find the determinant of matrix B.

$$\det B = \det\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix} = (-3)(-2) - (1)(5)$$

$$= (6) - (5)$$

$$\det B = 1$$

Secondly, substitute the value of **det B = 1** into the formula, and then reorganize the entries of matrix B to conform with the formula.

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix}$$

$$= 1\begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-2) & (1)(-1) \\ (1)(-5) & (1)(-3) \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix}$$

Example 3: Find the inverse of the matrix below, if it exists.

$$D = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Solution :

This fraction becomes undefined!

$$D^{-1} = \frac{1}{\det D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If det $D = 0$ ...

Let's go back to the problem to find the determinant of matrix D.

$$\begin{aligned} \det D &= \det \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \\ &= (4)(1) - (2)(2) \\ &= 4 - 4 \\ \det D &= 0 \end{aligned}$$

Therefore, **the inverse of matrix D does not exist** because the determinant of D equals zero. This is our final answer!

How to Find the Inverse of 3 x 3 Matrix?

- Compute the determinant of the given matrix.
- Calculate the determinant of 2×2 minor matrices.
- Formulate the matrix of cofactors.

- Take the transpose of the matrix of cofactors.
- Finally, divide each term of the adjugate matrix by the determinant.

$$A^{-1} = \frac{1}{det\,(A)}\ adj\,(A)$$

Definition of Adjoint of a Matrix

The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where Aij is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by **adj A**.

$$\text{Thus if } A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

$$\text{Adj.A} = \text{transpose of} \begin{bmatrix} A_{11} & A_{12} & \cdots\cdots & A_{1n} \\ A_{21} & A_{22} & \cdots\cdots & A_{2n} \\ A_{n1} & A_{n2} & \cdots\cdots & A_{nn} \end{bmatrix}$$

Inverse Matrix Formula

First, find the determinant of 3 × 3Matrix and then find it's minor, cofactors and adjoint and insert the results in the Inverse Matrix formula given below:

$$A^{-1} = \frac{adj\ A}{|A|}$$

Where $|A| \neq 0$

Inverse of a 3 x 3 Matrix Example

Example 1 : Find the inverse of the matrix.

$$\text{(ii)} \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Solution :

In order to find inverse of a matrix, first we have to find |A|.

$$|A| = 5(25 - 1) - 1(5 - 1) + 1(1 - 5)$$

$$= 5(24) - 1(4) + 1(-4)$$

$$= 120 - 4 - 4$$

$$= 112$$

Since $|A| = 112 \neq 0$, it is non singular matrix. A^{-1} exists.

$$a_{ij} = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$a_{ij} = \begin{pmatrix} +(25 - 1) & -(5 - 1) & +(1 - 5) \\ -(5 - 1) & +(25 - 1) & -(5 - 1) \\ +(1 - 5) & -(5 - 1) & +(5 - 1) \end{pmatrix}$$

$$a_{ij} = \begin{pmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 4 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{112} \begin{pmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{28} \begin{pmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

Example 2: Find the inverse of the matrix.

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution :

In order to find inverse of a matrix, first we have to find $|A|$.

$$|A| = 2(8 - 7) - 3(6 - 3) + 1(21 - 12)$$

$$= 2(1) - 3(3) + 1(9)$$

$$= 2 - 9 + 9$$

$$= 2$$

Since $|A| = 2 \neq 0$, it is non singular matrix. A^{-1} exists.

$$a_{ij} = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}$$

$$a_{ij} = \begin{pmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{pmatrix}$$

$$a_{ij} = \begin{pmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{pmatrix}$$