9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to -7} (2x + 5)$

12. $\lim_{x\to 2} (-x^2 + 5x - 2)$

13. $\lim_{t\to 6} 8(t - 5)(t - 7)$

14. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to 2} \dfrac{x + 3}{x + 6}$

16. $\lim_{s\to 2/3} 3s(2s - 1)$

17. $\lim_{x\to -1} 3(2x - 1)^2$

18. $\lim_{y\to 2} \dfrac{y + 2}{y^2 + 5y + 6}$

19. $\lim_{y\to -3} (5 - y)^{4/3}$

20. $\lim_{z\to 0} (2z - 8)^{1/3}$

21. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h + 1} + 1}$

22. $\lim_{h\to 0} \dfrac{\sqrt{5h + 4} - 2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to 5} \dfrac{x - 5}{x^2 - 25}$

24. $\lim_{x\to -3} \dfrac{x + 3}{x^2 + 4x + 3}$

25. $\lim_{x\to -5} \dfrac{x^2 + 3x - 10}{x + 5}$

26. $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x - 2}$

27. $\lim_{t\to 1} \dfrac{t^2 + t - 2}{t^2 - 1}$

28. $\lim_{t\to -1} \dfrac{t^2 + 3t + 2}{t^2 - t - 2}$

29. $\lim_{x\to -2} \dfrac{-2x - 4}{x^3 + 2x^2}$

30. $\lim_{y\to 0} \dfrac{5y^3 + 8y^2}{3y^4 - 16y^2}$

31. $\lim_{x\to 1} \dfrac{\frac{1}{x} - 1}{x - 1}$

32. $\lim_{x\to 0} \dfrac{\frac{1}{x - 1} + \frac{1}{x + 1}}{x}$

33. $\lim_{u\to 1} \dfrac{u^4 - 1}{u^3 - 1}$

34. $\lim_{v\to 2} \dfrac{v^3 - 8}{v^4 - 16}$

35. $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

36. $\lim_{x\to 4} \dfrac{4x - x^2}{2 - \sqrt{x}}$

37. $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x + 3} - 2}$

38. $\lim_{x\to -1} \dfrac{\sqrt{x^2 + 8} - 3}{x + 1}$

39. $\lim_{x\to 2} \dfrac{\sqrt{x^2 + 12} - 4}{x - 2}$

40. $\lim_{x\to -2} \dfrac{x + 2}{\sqrt{x^2 + 5} - 3}$

41. $\lim_{x\to -3} \dfrac{2 - \sqrt{x^2 - 5}}{x + 3}$

42. $\lim_{x\to 4} \dfrac{4 - x}{5 - \sqrt{x^2 + 9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim_{x\to 0} (2 \sin x - 1)$

44. $\lim_{x\to 0} \sin^2 x$

45. $\lim_{x\to 0} \sec x$

46. $\lim_{x\to 0} \tan x$

47. $\lim_{x\to 0} \dfrac{1 + x + \sin x}{3 \cos x}$

48. $\lim_{x\to 0} (x^2 - 1)(2 - \cos x)$

49. $\lim_{x\to -\pi} \sqrt{x + 4} \cos (x + \pi)$

50. $\lim_{x\to 0} \sqrt{7 + \sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0} \frac{2f(x) - g(x)}{(f(x) + 7)^{2/3}} = \frac{\lim_{x\to 0} (2f(x) - g(x))}{\lim_{x\to 0} (f(x) + 7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} \left(f(x) + 7\right)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2 \lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1) - (-5)}{(1 + 7)^{2/3}} = \frac{7}{4}$$

52. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))} = \frac{\lim_{x\to 1} \sqrt{5h(x)}}{\lim_{x\to 1} (p(x)(4 - r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} \left(4 - r(x)\right)\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5 \lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4 - 2)} = \frac{5}{2}$$

53. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim_{x\to c} f(x)g(x)$

b. $\lim_{x\to c} 2f(x)g(x)$

c. $\lim_{x\to c} (f(x) + 3g(x))$

d. $\lim_{x\to c} \dfrac{f(x)}{f(x) - g(x)}$

54. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

a. $\lim_{x\to 4} (g(x) + 3)$

b. $\lim_{x\to 4} xf(x)$

c. $\lim_{x\to 4} (g(x))^2$

d. $\lim_{x\to 4} \dfrac{g(x)}{f(x) - 1}$

55. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim_{x\to b} (f(x) + g(x))$

b. $\lim_{x\to b} f(x) \cdot g(x)$

c. $\lim_{x\to b} 4g(x)$

d. $\lim_{x\to b} f(x)/g(x)$

56. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

a. $\lim_{x\to -2} (p(x) + r(x) + s(x))$

b. $\lim_{x\to -2} p(x) \cdot r(x) \cdot s(x)$

c. $\lim_{x\to -2} (-4p(x) + 5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x) = x^2, \quad x = 1$

58. $f(x) = x^2, \quad x = -2$

59. $f(x) = 3x - 4, \quad x = 2$

60. $f(x) = 1/x, \quad x = -2$

61. $f(x) = \sqrt{x}, \quad x = 7$

62. $f(x) = \sqrt{3x+1}, \quad x = 0$

Using the Sandwich Theorem

63. If $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, find $\lim_{x \to 0} f(x)$.

64. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x \to 0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y = 1 - (x^2/6)$, $y = (x \sin x)/(2 - 2\cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

66. a. Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 10.9.) What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Estimating Limits

T You will find a graphing calculator useful for Exercises 67–74.

67. Let $f(x) = (x^2 - 9)/(x + 3)$.

a. Make a table of the values of f at the points $x = -3.1$, -3.01, -3.001, and so on as far as your calculator can go. Then estimate $\lim_{x \to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $x_0 = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x \to -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41, 1.414$, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \to \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $x_0 = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x \to \sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01, -6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x \to -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $x_0 = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x \to 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$.

a. Make tables of the values of f at values of x that approach $x_0 = -1$ from above and below. Then estimate $\lim_{x \to -1} f(x)$.

b. Support your conclusion in part (a) by graphing f near $x_0 = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x \to -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $x_0 = -2$ from above and below. Then estimate $\lim_{x \to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $x_0 = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x \to -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

Theory and Examples

75. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

76. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x\to 2} f(x) = 0$? Give reasons for your answers.

77. If $\lim_{x\to 4} \dfrac{f(x) - 5}{x - 2} = 1$, find $\lim_{x\to 4} f(x)$.

78. If $\lim_{x\to -2} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to -2} f(x)$ **b.** $\lim_{x\to -2} \dfrac{f(x)}{x}$

79. a. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 3$, find $\lim_{x\to 2} f(x)$.

b. If $\lim_{x\to 2} \dfrac{f(x) - 5}{x - 2} = 4$, find $\lim_{x\to 2} f(x)$.

80. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to 0} \dfrac{f(x)}{x}$

T **81. a.** Graph $g(x) = x \sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **82. a.** Graph $h(x) = x^2 \cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 83–88, use a CAS to perform the following steps:

a. Plot the function near the point x_0 being approached.

b. From your plot guess the value of the limit.

83. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

84. $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x + 1)^2}$

85. $\lim_{x\to 0} \dfrac{\sqrt[3]{1 + x} - 1}{x}$

86. $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2 + 7} - 4}$

87. $\lim_{x\to 0} \dfrac{1 - \cos x}{x \sin x}$

88. $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. We replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a precise definition, we can prove the limit properties given in the preceding section and establish many important limits.

To show that the limit of $f(x)$ as $x \to x_0$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to x_0. Let us see what this would require if we specified the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x_0 = 4$. Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x_0 = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

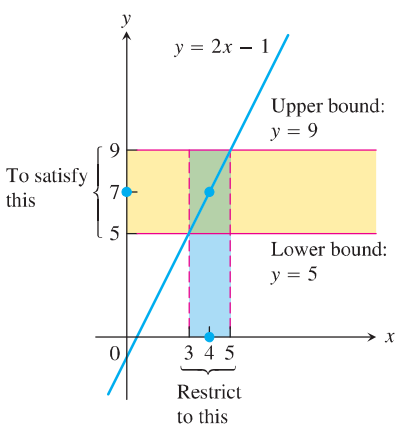


FIGURE 2.15 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned} |2x - 8| &< 2 \\ -2 < 2x - 8 &< 2 \\ 6 < 2x &< 10 \\ 3 < x &< 5 \\ -1 < x - 4 &< 1. \end{aligned}$$

Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Figure 2.15). ■

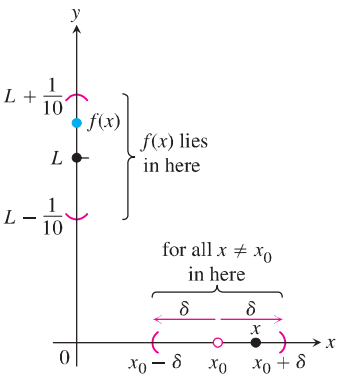


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

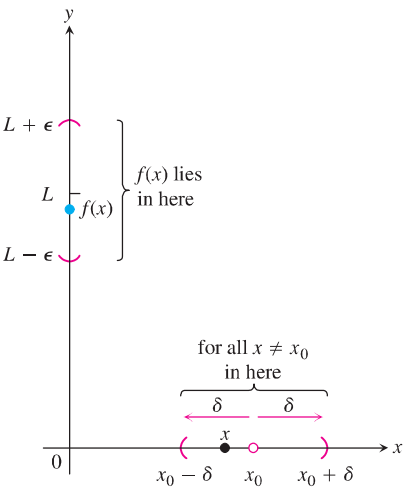


FIGURE 2.17 The relation of δ and ϵ in the definition of limit.

In the previous example we determined how close x must be to a particular value x_0 to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L. To show that the limit of $f(x)$ as $x \to x_0$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how small, by holding x close enough to x_0.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches x_0 (without taking on the value of x_0 itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of x_0 (Figure 2.16). But that in itself is not enough, because as x continues on its course toward x_0, what is to prevent $f(x)$ from jittering about within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L?

We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100{,}000$. Each time, we find a new δ-interval about x_0 so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ jitters away from L at some stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ϵ-challenges to prove that the limit does not exist or, more precisely, that there is room for doubt. The scholar answers every challenge with a δ-interval around x_0 that keeps the function values within ϵ of L.

How do we stop this seemingly endless series of challenges and responses? By proving that for every error tolerance ϵ that the challenger can produce, we can find, calculate, or conjure a matching distance δ that keeps x "close enough" to x_0 to keep $f(x)$ within that tolerance of L (Figure 2.17). This leads us to the precise definition of a limit.

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

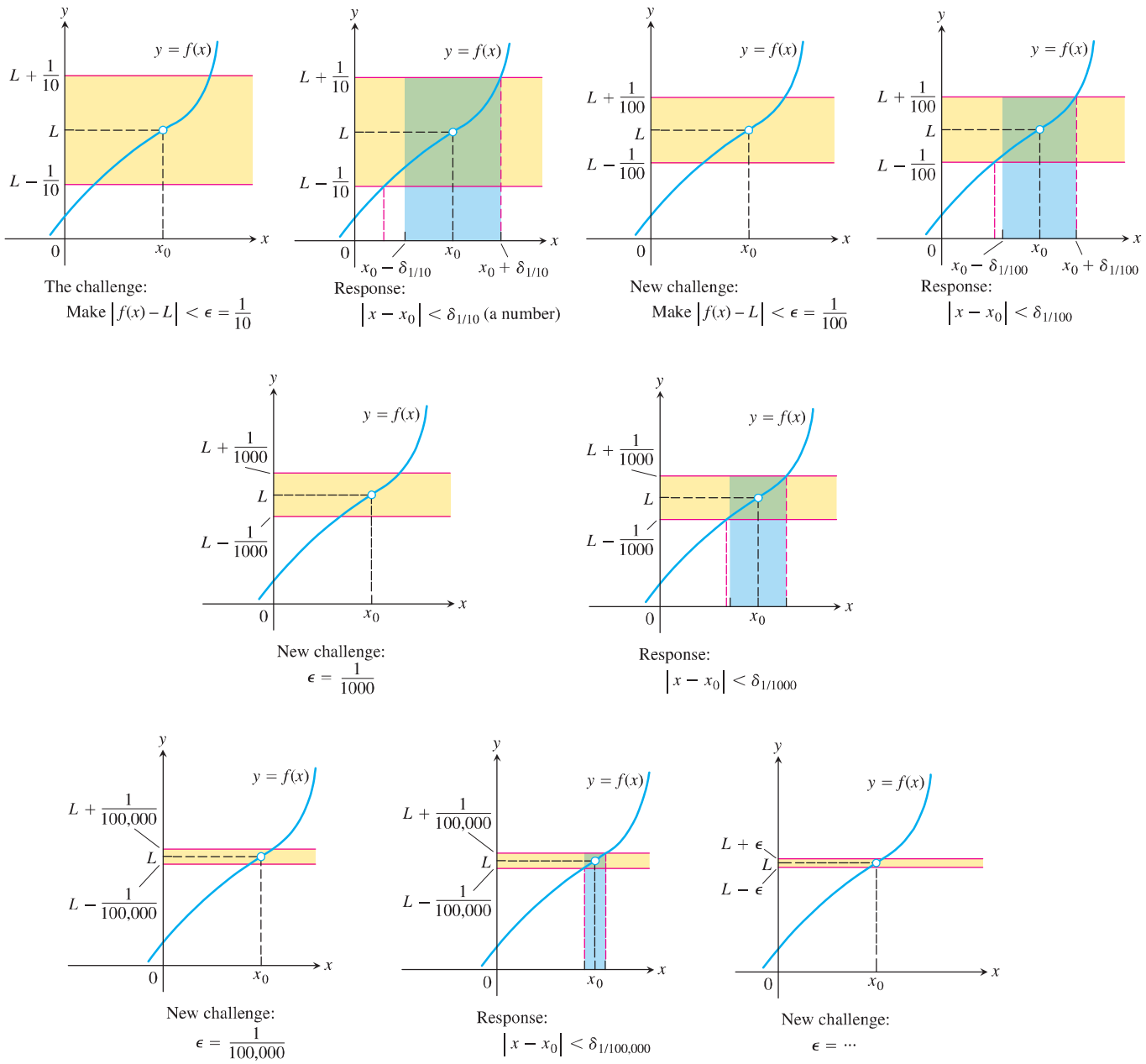
if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

One way to think about the definition is to suppose we are machining a generator shaft to a close tolerance. We may try for diameter L, but since nothing is perfect, we must be satisfied with a diameter $f(x)$ somewhere between $L - \epsilon$ and $L + \epsilon$. The δ is the measure of how accurate our control setting for x must be to guarantee this degree of accuracy in the diameter of the shaft. Notice that as the tolerance for error becomes stricter, we may have to adjust δ. That is, the value of δ, how tight our control setting must be, depends on the value of ϵ, the error tolerance.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it enables us to verify that a suspected limit is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified.



EXAMPLE 2 Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

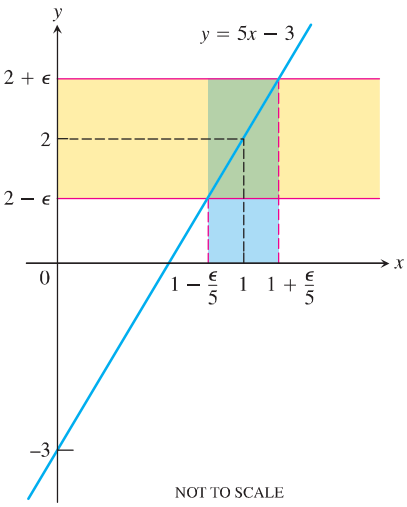


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x\to 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ■

EXAMPLE 3 Prove the following results presented graphically in Section 2.2.

(a) $\lim_{x\to x_0} x = x_0$ **(b)** $\lim_{x\to x_0} k = k$ (k constant)

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |x - x_0| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number (Figure 2.19). This proves that $\lim_{x\to x_0} x = x_0$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.20). This proves that $\lim_{x\to x_0} k = k$. ■

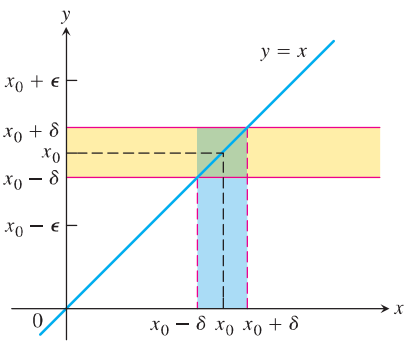


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - x_0| < \delta$ will guarantee $|f(x) - x_0| < \epsilon$ whenever $\delta \le \epsilon$ (Example 3a).

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about x_0 for which $|f(x) - L|$ was less than ϵ was symmetric about x_0 and we could take δ to be half the length of that interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from x_0 to the interval's *nearer* endpoint.

EXAMPLE 4 For the limit $\lim_{x\to 5}\sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

Solution We organize the search into two steps, as discussed below.

1. *Solve the inequality* $|\sqrt{x - 1} - 2| < 1$ *to find an interval containing* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$\begin{aligned} |\sqrt{x - 1} - 2| &< 1 \\ -1 < \sqrt{x - 1} - 2 &< 1 \\ 1 < \sqrt{x - 1} &< 3 \\ 1 < x - 1 &< 9 \\ 2 < x &< 10 \end{aligned}$$

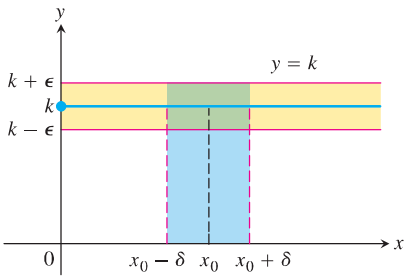


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \epsilon$ for any positive δ (Example 3b).