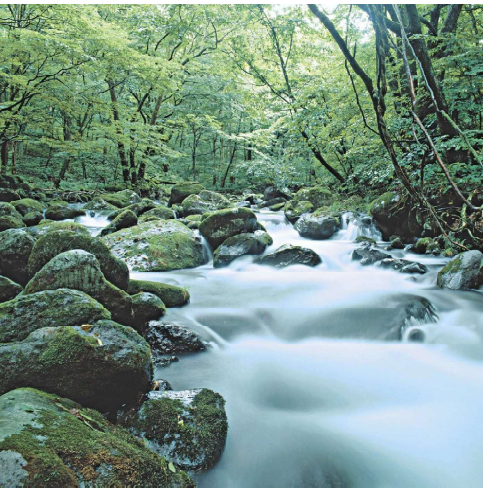


6

Applications of Definite Integrals

OVERVIEW In Chapter 5 we saw that a continuous function over a closed interval has a definite integral, which is the limit of any Riemann sum for the function. We proved that we could evaluate definite integrals using the Fundamental Theorem of Calculus. We also found that the area under a curve and the area between two curves could be computed as definite integrals.

In this chapter we extend the applications of definite integrals to finding volumes, lengths of plane curves, and areas of surfaces of revolution. We also use integrals to solve physical problems involving the work done by a force, the fluid force against a planar wall, and the location of an object's center of mass.

6.1 Volumes Using Cross-Sections

In this section we define volumes of solids using the areas of their cross-sections. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1). We present three different methods for obtaining the cross-sections appropriate to finding the volume of a particular solid: the method of slicing, the disk method, and the washer method.

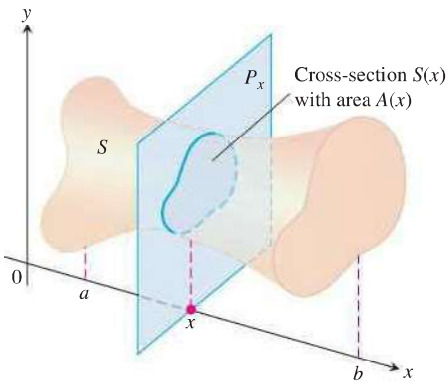


FIGURE 6.1 A cross-section $S(x)$ of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylinders, like the one in Figure 6.1. If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $S(x)$ of area $A(x)$, and A is a continuous function of x, we can define and calculate the volume of the solid S as the definite integral of $A(x)$. We now show how this integral is obtained by the **method of slicing**.

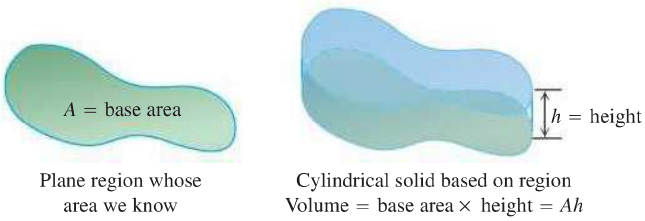


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

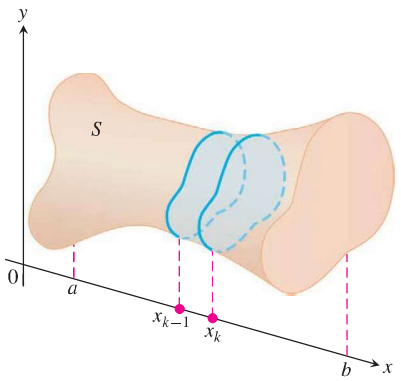


FIGURE 6.3 A typical thin slab in the solid S.

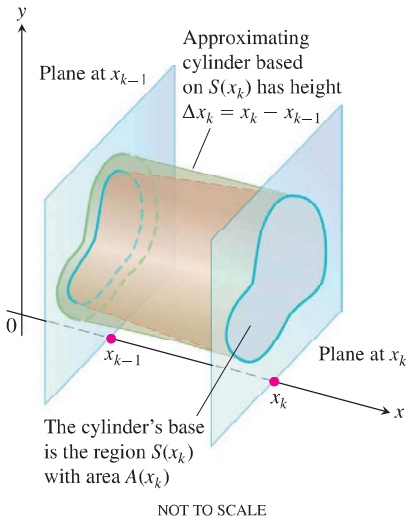


FIGURE 6.4 The solid thin slab in Figure 6.3 is shown enlarged here. It is approximated by the cylindrical solid with base $S(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

Slicing by Parallel Planes

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. The planes P_{x_k}, perpendicular to the x-axis at the partition points, slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\,\Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\,\Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero. Taking a partition of $[a, b]$ into n subintervals with $\|P\| \to 0$ gives

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\,dx.$$

So we define the limiting definite integral of the Riemann sum to be the volume of the solid S.

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\,dx.$$

This definition applies whenever $A(x)$ is integrable, and in particular when it is continuous. To apply the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch.* We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5).

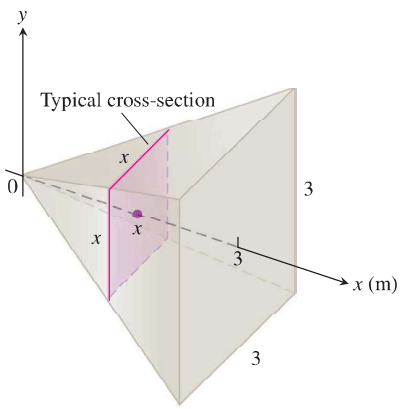


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

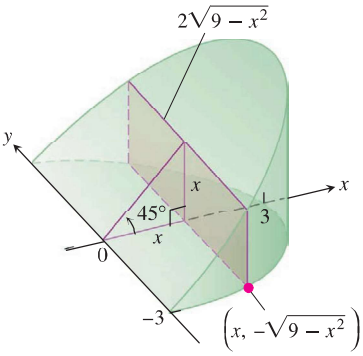


FIGURE 6.6 The wedge of Example 2, sliced perpendicular to the x-axis. The cross-sections are rectangles.

2. *A formula for $A(x)$.* The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration.* The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume:*

$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = 9\text{ m}^3.$$

■

EXAMPLE 2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.6). The base of the wedge in the figure is the semi-circle with $x \geq 0$ that is cut from the circle $x^2 + y^2 = 9$ by the 45° plane when it intersects the y-axis. For any x in the interval $[0, 3]$, the y-values in this semi-circular base vary from $y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$. When we slice through the wedge by a plane perpendicular to the x-axis, we obtain a cross-section at x which is a rectangle of height x whose width extends across the semi-circular base. The area of this cross-section is

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)\left(2\sqrt{9 - x^2}\right) \\ &= 2x\sqrt{9 - x^2}\,. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V = \int_a^b A(x)\,dx &= \int_0^3 2x\sqrt{9 - x^2}\,dx \\ &= -\frac{2}{3}(9 - x^2)^{3/2}\Big]_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

■

EXAMPLE 3 Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.7). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids.

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

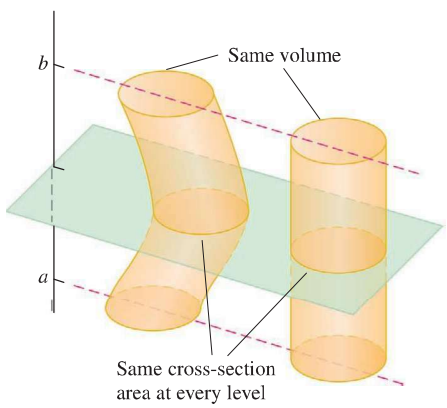


FIGURE 6.7 *Cavalieri's principle:* These solids have the same volume, which can be illustrated with stacks of coins.

■

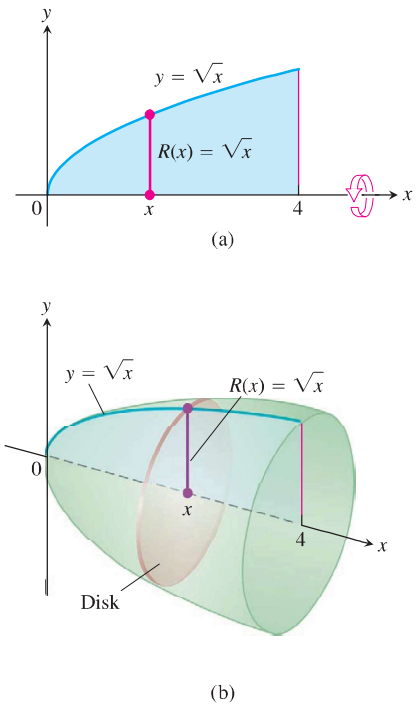


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

Solids of Revolution: The Disk Method

The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume in this case gives

Volume by Disks for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad \text{Radius } R(x) = \sqrt{x} \text{ for rotation around } x\text{-axis} \\ &= \pi\int_0^4 x\,dx = \pi\left.\frac{x^2}{2}\right]_0^4 = \pi\frac{(4)^2}{2} = 8\pi. \end{aligned}$$

■

EXAMPLE 5 The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2). \qquad R(x) = \sqrt{a^2 - x^2} \text{ for rotation around } x\text{-axis}$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

■

The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

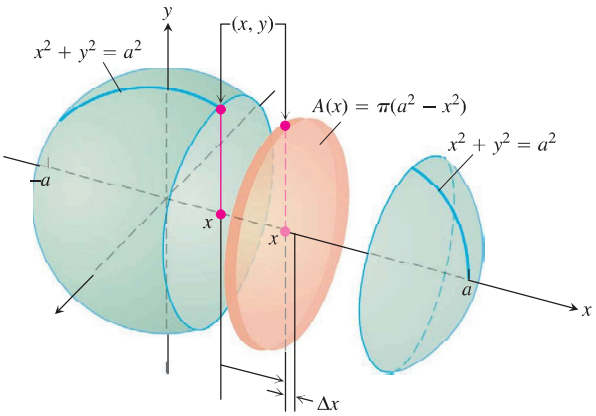


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$V = \int_1^4 \pi[R(x)]^2\,dx$$

$$= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \qquad \text{Radius } R(x) = \sqrt{x} - 1 \text{ for rotation around } y = 1$$

$$= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \qquad \text{Expand integrand.}$$

$$= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. \qquad \text{Integrate.}$$

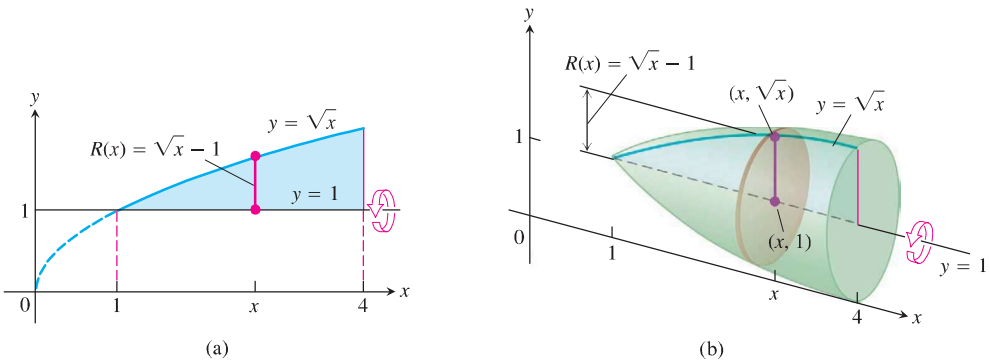


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6. ■

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2,$$

and the definition of volume gives

Volume by Disks for Rotation About the y-axis

$$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$

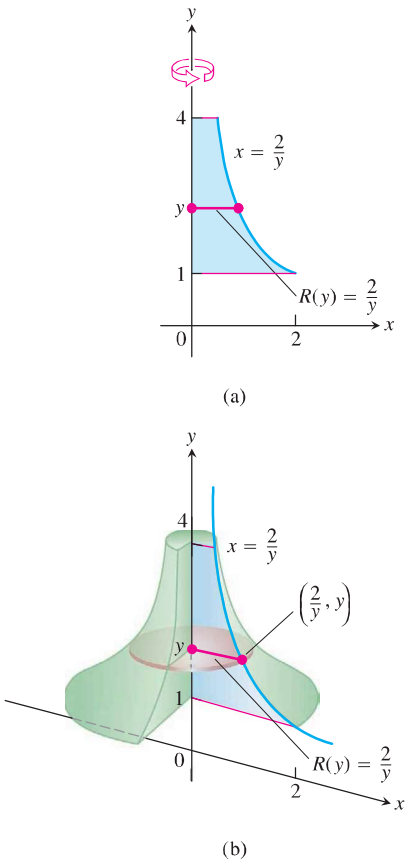


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

EXAMPLE 7 Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$V = \int_1^4 \pi[R(y)]^2\,dy$$

$$= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \qquad \text{Radius } R(y) = \frac{2}{y} \text{ for rotation around } y\text{-axis}$$

$$= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] = 3\pi.$$

■

EXAMPLE 8 Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$ and have y-coordinates from $y = -\sqrt{2}$ to $y = \sqrt{2}$. The volume is

$$V = \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\,dy \qquad y = \pm\sqrt{2} \text{ when } x = 3$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\,dy \qquad \text{Radius } R(y) = 3 - (y^2 + 1) \text{ for rotation around axis } x = 3$$

$$= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\,dy \qquad \text{Expand integrand.}$$

$$= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \qquad \text{Integrate.}$$

$$= \frac{64\pi\sqrt{2}}{15}.$$

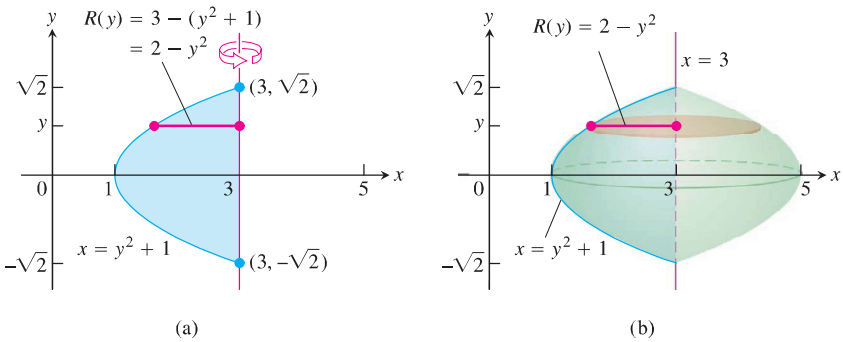


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8. ■

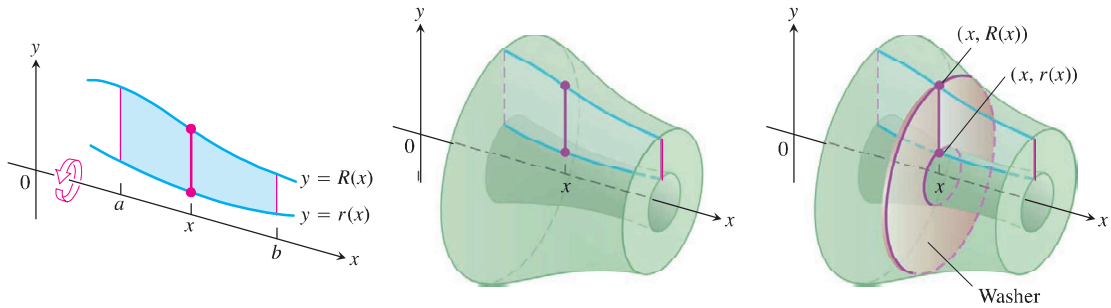


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are *washers* (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$
$$\text{Inner radius:} \quad r(x)$$

The washer's area is

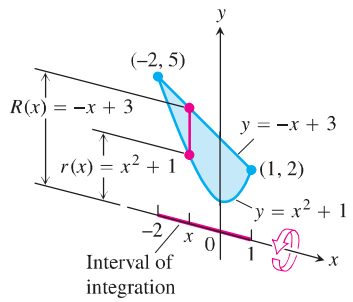
$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume in this case gives

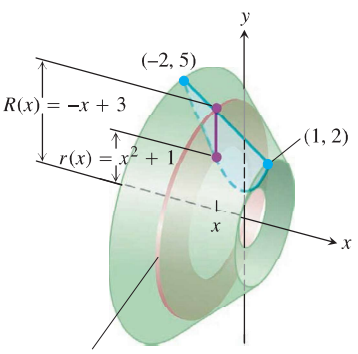
Volume by Washers for Rotation About the x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a thin slab of the solid resembles a circular washer of outer radius $R(x)$ and inner radius $r(x)$.



(a)



Washer cross-section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

(b)

FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

EXAMPLE 9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution We use the four steps for calculating the volume of a solid as discussed early in this section.

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14a).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

   These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\text{Outer radius:} \quad R(x) = -x + 3$$
$$\text{Inner radius:} \quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

Limits of integration

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\, dx && \text{Rotation around } x\text{-axis} \\ &= \int_{-2}^{1} \pi((-x + 3)^2 - (x^2 + 1)^2)\, dx && \text{Values from Steps 2 and 3} \\ &= \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\, dx && \text{Simplify algebraically.} \\ &= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$

■

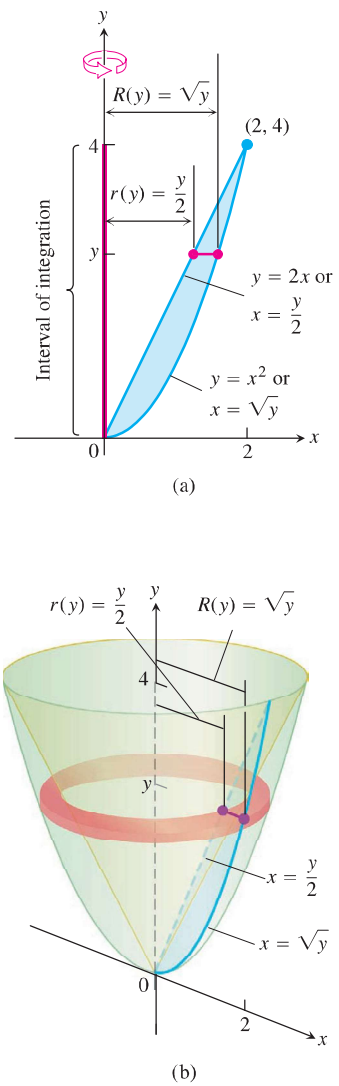


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x. In this situation the line segment sweeping out a typical washer is perpendicular to the y-axis (the axis of revolution), and the outer and inner radii of the washer are functions of y.

EXAMPLE 10 The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\, dy && \text{Rotation around } y\text{-axis} \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy && \text{Substitute for radii and limits of integration.} \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$

■

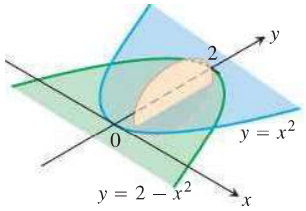
Exercises 6.1

Volumes by Slicing

Find the volumes of the solids in Exercises 1–10.

1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

2. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

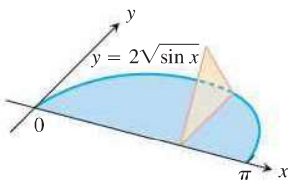


3. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

5. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

   a. equilateral triangles with bases running from the x-axis to the curve as shown in the accompanying figure.



   b. squares with bases running from the x-axis to the curve.

6. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x-axis are

   a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

   b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

7. The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the x-axis are

   a. rectangles of height 10.

   b. rectangles of perimeter 20.

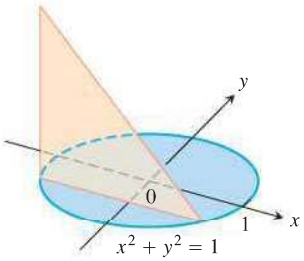
8. The base of a solid is the region bounded by the graphs of $y = \sqrt{x}$ and $y = x/2$. The cross-sections perpendicular to the x-axis are

   a. isosceles triangles of height 6.

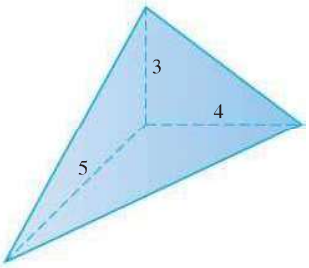
   b. semi-circles with diameters running across the base of the solid.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

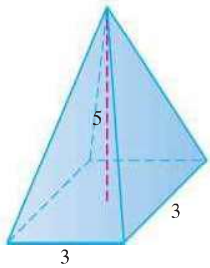
10. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.



11. Find the volume of the given tetrahedron. (Hint: Consider slices perpendicular to one of the labeled edges.)



12. Find the volume of the given pyramid, which has a square base of area 9 and height 5.

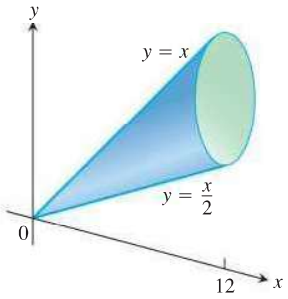


13. **A twisted solid** A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.

   a. Find the volume of the column.

   b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.

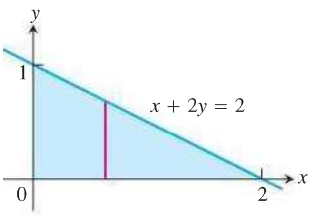
14. Cavalieri's principle A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



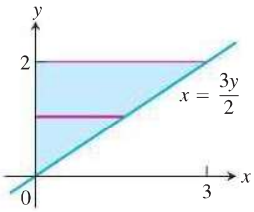
Volumes by the Disk Method

In Exercises 15–18, find the volume of the solid generated by revolving the shaded region about the given axis.

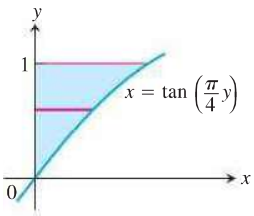
15. About the x-axis



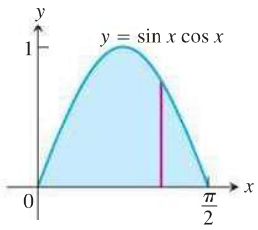
16. About the y-axis



17. About the y-axis



18. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 19–24 about the x-axis.

19. $y = x^2, \quad y = 0, \quad x = 2$

20. $y = x^3, \quad y = 0, \quad x = 2$

21. $y = \sqrt{9 - x^2}, \quad y = 0$

22. $y = x - x^2, \quad y = 0$

23. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

24. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 25 and 26, find the volume of the solid generated by revolving the region about the given line.

25. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

26. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2 \sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 27–32 about the y-axis.

27. The region enclosed by $x = \sqrt{5}\, y^2, \quad x = 0, \quad y = -1, \quad y = 1$

28. The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

29. The region enclosed by $x = \sqrt{2 \sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

30. The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

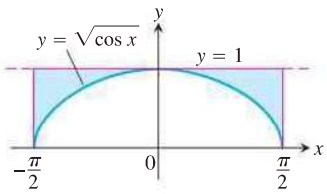
31. $x = 2/(y + 1), \quad x = 0, \quad y = 0, \quad y = 3$

32. $x = \sqrt{2y}/(y^2 + 1), \quad x = 0, \quad y = 1$

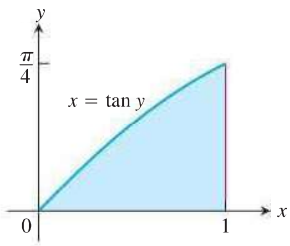
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 33 and 34 about the indicated axes.

33. The x-axis



34. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 35–40 about the x-axis.

35. $y = x, \quad y = 1, \quad x = 0$

36. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

37. $y = x^2 + 1, \quad y = x + 3$

38. $y = 4 - x^2, \quad y = 2 - x$

39. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

40. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 41–44, find the volume of the solid generated by revolving each region about the y-axis.

41. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

42. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

43. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

44. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 45 and 46, find the volume of the solid generated by revolving each region about the given axis.

45. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

46. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

47. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis.

c. the line $y = 2$. **d.** the line $x = 4$.

48. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

49. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a. the line $y = 1$. **b.** the line $y = 2$.

c. the line $y = -1$.

50. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about

a. the x-axis. **b.** the y-axis.

Theory and Applications

51. The volume of a torus The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

52. Volume of a bowl A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

a. Find the volume of the bowl.

b. Related rates If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

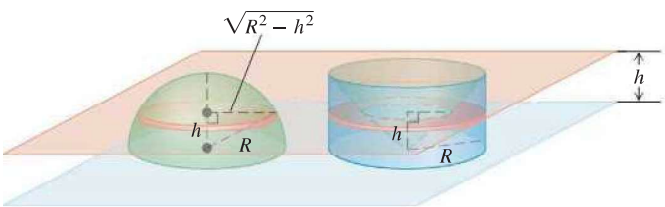
53. Volume of a bowl

a. A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

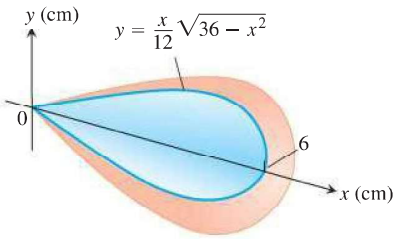
b. Related rates Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of $0.2\ \text{m}^3/\text{sec}$. How fast is the water level in the bowl rising when the water is 4 m deep?

54. Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

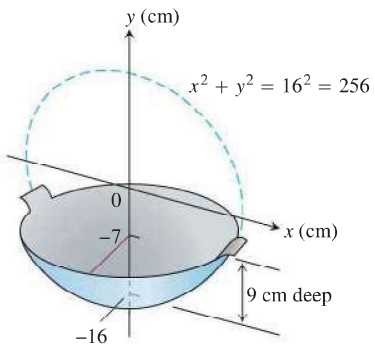
55. Volume of a hemisphere Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed, as suggested by the accompanying figure.



56. Designing a plumb bob Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs $8.5\ \text{g/cm}^3$, how much will the plumb bob weigh (to the nearest gram)?



57. Designing a wok You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $(1\ \text{L} = 1000\ \text{cm}^3.)$

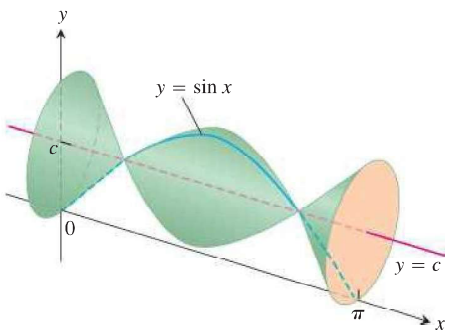


58. Max-min The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in the accompanying figure.

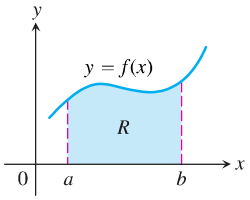
a. Find the value of c that minimizes the volume of the solid. What is the minimum volume?

b. What value of c in $[0, 1]$ maximizes the volume of the solid?

T **c.** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.



59. Consider the region R bounded by the graphs of $y = f(x) > 0$, $x = a > 0$, $x = b > a$, and $y = 0$ (see accomanying figure). If the volume of the solid formed by revolving R about the x-axis is 4π, and the volume of the solid formed by revolving R about the line $y = -1$ is 8π, find the area of R.



60. Consider the region R given in Exercise 59. If the volume of the solid formed by revolving R around the x-axis is 6π, and the volume of the solid formed by revolving R around the line $y = -2$ is 10π, find the area of R.

6.2 Volumes Using Cylindrical Shells

In Section 6.1 we defined the volume of a solid as the definite integral $V = \int_a^b A(x)\,dx$, where $A(x)$ is an integrable cross-sectional area of the solid from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. However, this method of slicing is sometimes awkward to apply, as we will illustrate in our first example. To overcome this difficulty, we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way.

Slicing with Cylinders

Suppose we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid so that the axis of each cylinder is parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This slab interpretation allows us to apply the same integral definition for volume as before. The following example provides some insight before we derive the general method.

EXAMPLE 1 The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid (Figure 6.16). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x-values of the left and right sides of the parabola in Figure 6.16a in terms of y. (These x-values are the inner and outer radii for a typical washer, requiring us to solve $y = 3x - x^2$ for x, which leads to complicated formulas.) Instead of rotating a horizontal strip of thickness Δy, we rotate a *vertical strip* of thickness Δx. This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip and of thickness Δx. An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.17. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole. We then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of

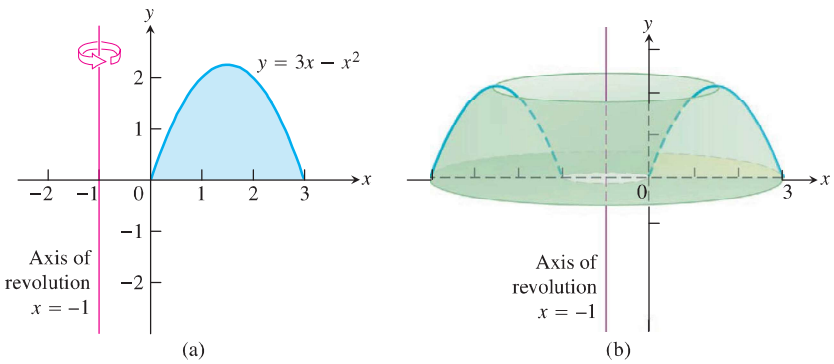


FIGURE 6.16 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

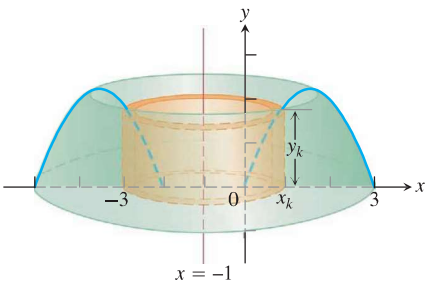


FIGURE 6.17 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx_k about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.16a).

Each slice is sitting over a subinterval of the x-axis of length (width) Δx_k. Its radius is approximately $(1 + x_k)$, and its height is approximately $3x_k - x_k^2$. If we unroll the cylinder at x_k and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx_k (Figure 6.18). The outer circumference of the kth cylinder is $2\pi \cdot \text{radius} = 2\pi(1 + x_k)$, and this is the length of the rolled-out rectangular slab. Its volume is approximated by that of a rectangular solid,

$$\begin{aligned}\Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness}\\ &= 2\pi(1 + x_k) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x_k.\end{aligned}$$

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k^2\right) \Delta x_k.$$

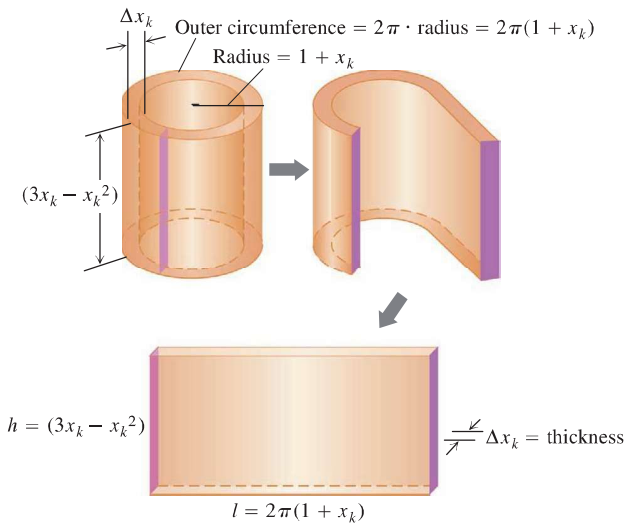


FIGURE 6.18 Cutting and unrolling a cylindrical shell gives a nearly rectangular solid (Example 1).

Taking the limit as the thickness $\Delta x_k \to 0$ and $n \to \infty$ gives the volume integral

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k^2\right) \Delta x_k \\ &= \int_0^3 2\pi(x + 1)(3x - x^2)\, dx \\ &= \int_0^3 2\pi(3x^2 + 3x - x^3 - x^2)\, dx \\ &= 2\pi \int_0^3 (2x^2 + 3x - x^3)\, dx \\ &= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{1}{4}x^4\right]_0^3 = \frac{45\pi}{2}. \end{aligned}$$

■

We now generalize the procedure used in Example 1.

The Shell Method

Suppose the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (Figure 6.19a). We assume $a \geq L$, so the vertical line may touch the region, but not pass through it. We generate a solid S by rotating this region about the vertical line L.

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$, and let c_k be the midpoint of the kth subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.19a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width $\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.19b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

The volume of a cylindrical shell of height h with inner radius r and outer radius R is

$$\pi R^2 h - \pi r^2 h = 2\pi\left(\frac{R + r}{2}\right)(h)(R - r)$$

$$\begin{aligned} \Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \end{aligned}$$

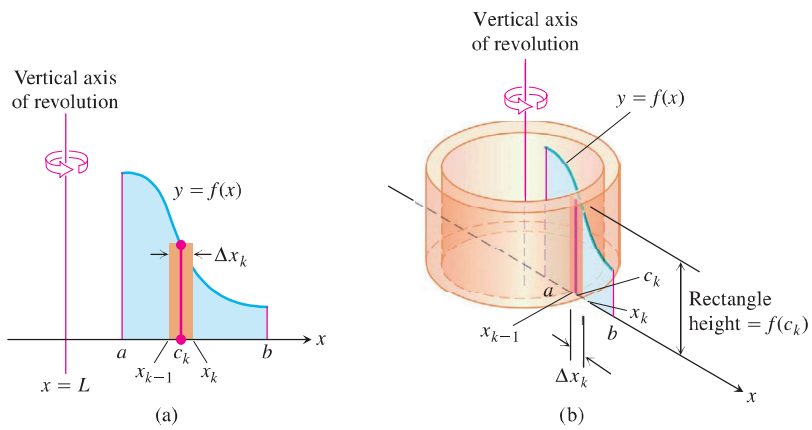


FIGURE 6.19 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles based on P:

$$V \approx \sum_{k=1}^{n} \Delta V_k.$$

The limit of this Riemann sum as each $\Delta x_k \to 0$ and $n \to \infty$ gives the volume of the solid as a definite integral:

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx.$$
$$= \int_a^b 2\pi(x - L)f(x)\, dx.$$

We refer to the variable of integration, here x, as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx.$$

EXAMPLE 2 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.20a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.20b, but you need not do that.)

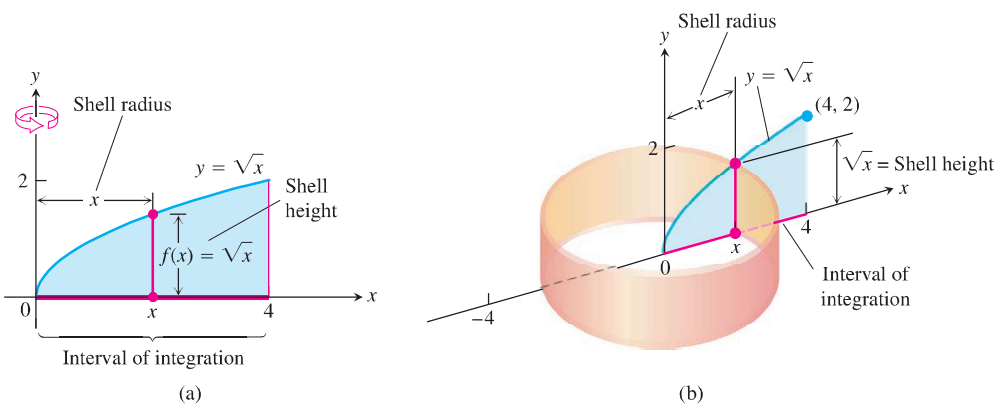


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx \\ &= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx \\ &= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left[\frac{2}{5} x^{5/2}\right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

■

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x's with y's.

EXAMPLE 3 The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid by the shell method.

Solution This is the solid whose volume was found by the disk method in Example 4 of Section 6.1. Now we find its volume by the shell method. First, sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.21). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= 2\pi \int_0^2 (4y - y^3)\, dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$

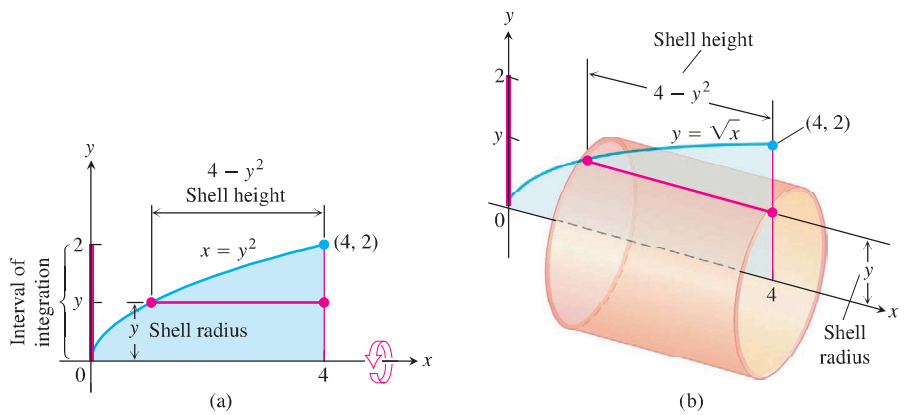


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy. ■

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find the limits of integration* for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

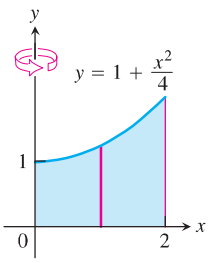
The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 37 and 38. (Exercise 60 in Section 7.1 outlines a proof.) Both volume formulas are actually special cases of a general volume formula we will look at when studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

Exercises 6.2

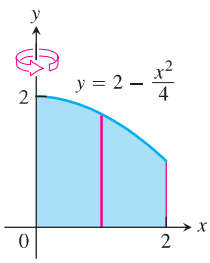
Revolution About the Axes

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

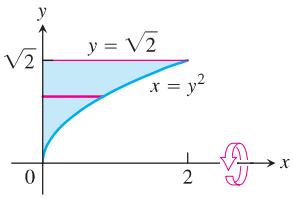
1.



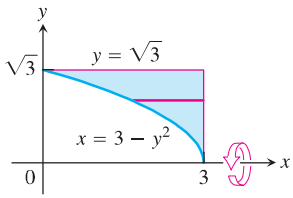
2.



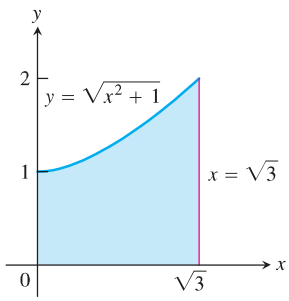
3.



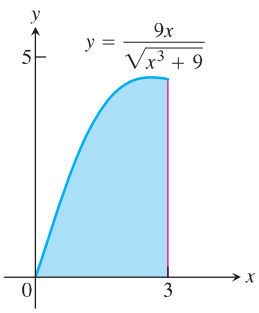
4.



5. The y-axis



6. The y-axis



Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–12 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0, \quad$ for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

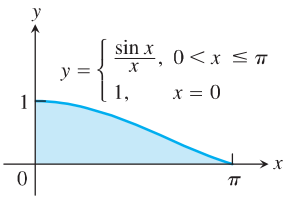
11. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

12. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0 \end{cases}$

a. Show that $xf(x) = \sin x$, $0 \le x \le \pi$.

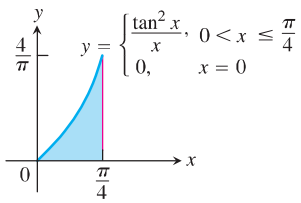
b. Find the volume of the solid generated by revolving the shaded region about the y-axis in the accompanying figure.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$

a. Show that $xg(x) = (\tan x)^2$, $0 \le x \le \pi/4$.

b. Find the volume of the solid generated by revolving the shaded region about the y-axis in the accompanying figure.



Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y}$, $x = -y$, $y = 2$

16. $x = y^2$, $x = -y$, $y = 2$, $y \ge 0$

17. $x = 2y - y^2$, $x = 0$

18. $x = 2y - y^2$, $x = y$

19. $y = |x|$, $y = 1$

20. $y = x$, $y = 2x$, $y = 2$

21. $y = \sqrt{x}$, $y = 0$, $y = x - 2$

22. $y = \sqrt{x}$, $y = 0$, $y = 2 - x$

Revolution About Horizontal and Vertical Lines

In Exercises 23–26, use the shell method to find the volumes of the solids generated by revolving the regions bounded by the given curves about the given lines.

23. $y = 3x$, $y = 0$, $x = 2$

a. The y-axis **b.** The line $x = 4$
c. The line $x = -1$ **d.** The x-axis
e. The line $y = 7$ **f.** The line $y = -2$

24. $y = x^3$, $y = 8$, $x = 0$

a. The y-axis **b.** The line $x = 3$
c. The line $x = -2$ **d.** The x-axis
e. The line $y = 8$ **f.** The line $y = -1$

25. $y = x + 2$, $y = x^2$

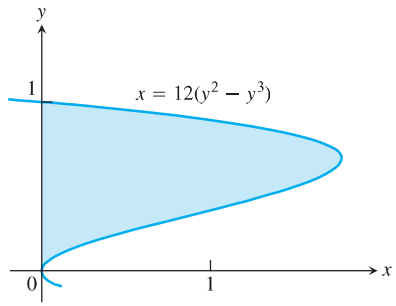
a. The line $x = 2$ **b.** The line $x = -1$
c. The x-axis **d.** The line $y = 4$

26. $y = x^4$, $y = 4 - 3x^2$

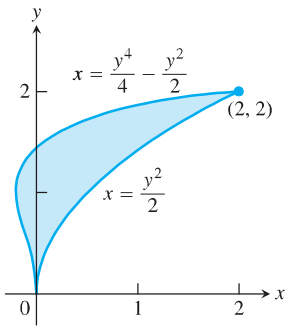
a. The line $x = 1$ **c.** The x-axis

In Exercises 27 and 28, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

27. **a.** The x-axis **b.** The line $y = 1$
c. The line $y = 8/5$ **b.** The line $y = -2/5$



28. **a.** The x-axis **b.** The line $y = 2$
c. The line $y = 5$ **d.** The line $y = -5/8$



Choosing the Washer Method or Shell Method

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. It may not be possible, however, to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead. Exercises 29 and 30 provide some insight.

29. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

a. the shell method. **b.** the washer method.

30. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

a. the x-axis using the washer method.

b. the y-axis using the shell method.

c. the line $x = 4$ using the shell method.

d. the line $y = 8$ using the washer method.